Special Issue: Resilient Distributed Estimator with Information Consensus for CPS Security

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***Abstract*—In this paper, we study the collaboratively estimating problem of a discrete-time LTI system with a time-varying undirected communication graph among sensors. The performance of resilient state estimators developed for cyber-physical systems (CPS) degenerates if the sensor measurements are compromised. To obtain robustness for the estimation, we propose Resilient Distributed Estimator with Information Consensus (RDEIC). RDEIC is a consensus-based resilient distributed algorithm that produces bounded state estimation errors with faulty sensors. Our algorithm converges to the true state in an attack-free scenario and it produces bounded estimation errors during an attack. The performance of the proposed algorithm is demonstrated with Matlab simulations.**

I. Introduction

Nowadays, the cyber-physical systems (CPS) have become ubiquitous in most aspects of human society, including healthcare, transportation and manufacturing, etc. The innumerable applications of CPS raise the interest of malicious parties. Sensors, which help to collect information for decision-making process, are distributed pervasively, and thus, easier to get accessed. Since sensors are typically resource-restricted, they are unlikely to be equipped with complex security mechanisms. These features make it relatively easier for adversaries to tamper with sensor readings, aka false data injection attacks (FDIA). FDIA on sensors have been studied extensively, including the GPS spoofing, LiDAR spoofing, and radar spoofing [1]. Therefore, ensuring the system resiliency to such attacks becomes necessary to the functioning of CPS.

In this paper, we investigate the distributed state estimation problem under FDIA. As the scale of CPS applications increases, e.g. connected vehicles, the computation cost of centralized and decentralized state estimation increases quadratically, making distributed algorithms preferred. Traditional distributed state estimation algorithms, such as Distributed Kalman Filter (DKF) [2], are not resilient to FDIA and the estimation errors are not bounded when an attack occurs [3]. To make the system resilient to attacks, various works have been proposed. Mitra et al. [4] developed a distributed observer by decomposing the linear system model into detectable and undetectable parts and estimating the two parts with a Luenberger observer and a consensus algorithm respectively. Dutta et al. [3] proposed a Resilient Distributed Kalman Filter (RDKF) by formulating the state estimation problem as an optimization problem. A compensation term was used to balance the physical dynamics residuals and the sensor measurement residuals. Nevertheless, these methods do not consider the dynamics of the network or the consensus among nodes. S. Wang [5] proposed a consensus extended Kalman filtering from a Bayesian perspective with the constraint that each node shared an agreement with neighbors. However, their method performed poorly if sensor attacks present.

To address the limitations, we propose Robust Distributed Estimator with Information Consensus (RDEIC), which can be applied for a time-varying communication network and consensus among nodes is considered. The contributions of the paper are listed as follows:

- Following the line of works [3], [6], RDEIC is developed based on the variants of DKF with two versions of the local loss functions being provided.
- RDEIC can be performed on both static and time-varying underlying communication graphs.
- Compared with DKF, RDEIC achieves better performance regardless of the presence of attacks.

The rest of the paper is organized as follows: In Section II, we present the formulation of the system, the attack model and the communication topology. In Section III, the distributed Kalman estimator and its variants are discussed and we provide two versions of local loss functions of each node. We present our RDEIC method and its ADMM solver in Section IV. Some numerical simulations are shown in Section V and conclusions are drawn in Section VI.

II. Problem Description

A. System Setup and Attack Model

We consider the following LTI system with n sensors,

$$x(k+1) = Ax(k) + Bw(k), y_i(k) = C_i x(k) + v_i(k), \tag{1}$$

where $x(k) \in \mathbb{R}^q$ is the state vector, $y_i \in \mathbb{R}^s, i \in [n]$ is the sensor measurement collected by the i-th sensor. The vectors $w(k) \sim \mathcal{N}(0,Q), v_i(k) \sim \mathcal{N}(0,R)$ are Gaussian noises for the system and the sensor, respectively. We assume that $w(k)$ and $v_i(k)$ are independent to each other. We also assume that Q and R are positive definite matrices. $A \in \mathbb{R}^{q\times q}$ is the system matrix, $B \in \mathbb{R}^{q\times q}$ is the noise matrix, and $C_i \in \mathbb{R}^{s\times q}$ is the observation matrix. We consider the attack taken place on sensor measurements as

$$y_i(k) = C_i x(k) + v_i(k) + a_i(k),$$

where $a_i(k)$ is the $i-$th attack vector at time k. When $a_i(k)$ is nonzero, we call the measurement $y_i(k)$ is compromised/corrupted. Denote $\mathcal{C} = [C_1 \cdots C_n]^\top$ as the congregation of $\{C_i\}_{i=1}^n$. The distributed estimator aims to asymptotically correct the estimates $\hat{x}_i(k)$ to the true state $x(k)$ by collecting information from neighbors for any node i. The pair (A, C_i) may not be detectable, and we only assume the pair $(A, \mathcal{C})$ is observable.

Information exchanges among the nodes are necessary for a distributed estimator. The information flow is described as a sequence of undirected graphs $\{\mathcal{G}(k)\}_{k=1}^\infty = \{(\mathcal{V}, \mathcal{E}(k)) \mid \mathcal{E}_k \subset \mathcal{V} \times \mathcal{V})\}$. When $\mathcal{G}(k) = \mathcal{G}$ for all $k \geq 1$, it is a static graph. To extend our consensus-based algorithm into a time-varying underlying communication graph, we additionally assume that the sequence $\{\mathcal{G}(k)\}_{k=1}^\infty$ is "jointly strongly-connected". The definition is given below:

Definition 1 (Jointly Strong-connectivity): There exists $T \in \mathbb{N}_+$ such that the union graph over the interval $[kT, (k+1)T]$, i.e. $\cup_{i=kT}^{(k+1)T} \mathcal{G}(i)$, is strongly connected, where $k \in \mathbb{N}$.

III. Distributed Kalman Filters and its Variants

To mitigate the impact of noises on the estimation accuracy, the classical Kalman filter (KF) contains *prediction* and *correction* stages. The prediction stage is represented in the following equation

$$\hat{x}(k+1|k) = A\hat{x}(k), P(k+1|k) = AP(k)A^\top + BQB^\top$$

where $P(k)$ is the estimation error covariance matrix and $P(k+1|k)$ is *a prior* error covariance matrix. The measurement updates of KF are given by

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k+1|k) + K(k)[y(k) - C\hat{x}(k+1|k)] \\ P(k+1) &= (I - K(k))CP(k+1|k) \\ K(k) &= P(k+1|k)C^\top[CP(k+1|k)C^\top + R]^{-1}, \end{aligned}$$

where $K(k)$ is the Kalman gain. According to [6], the classical KF can be rewritten from a Bayesian viewpoint as

$$\hat{x}(k+1) = P(k+1)\Big(C^\top R^{-1} y(k+1) + [P(k+1|k)]^{-1}\hat{x}(k+1|k)\Big)$$

The classical KF is centralised in the process of combining sensory data. Although the KF is ideally Bayesian optimal in terms of tracking performance, it has many practical limitations, such as scalability, communication delay and limited communication bandwidth [7]. To tackle these limitations, the distributed estimation was introduced, in which each sensor node performs estimation using the information from its own and connected neighbors.

Olfati-Saber [8], [9] proposed Kalman consensus filter (KCF) in which average consensus on local Kalman filtering were performed. S. Park *et al.* [10], [11] proposed a distributed observer where augmented states were used. A distributed Kalman filter (DKF) proposed by Dutta *et al.* [3] read

$$P_i(k+1|k) = AP_i(k)A^\top + BQB^\top \tag{2}$$

$$\begin{aligned} \hat{x}_i(k+1) = P_i(k+1)\Big(&\frac{1}{d_i}\sum_{j \in N_i}[P(k+1|k)]^{-1}A\hat{x}_j(k) \\ &+ C_i^\top R^{-1} y_i(k+1)\Big), \end{aligned} \tag{3}$$

where $N_i = \{i\} \cup \{j \mid j \text{ is a neighbor of } i\}$ and $d_i = |N_i|$ is the degree of node i. The local estimate $\hat{x}_i(k+1)$ evaluate the state $x(k+1)$ in the i-th node at time $k+1$ and $P_i(k+1)$ is the covariance matrix of the error $e_i(k+1) = \hat{x}_i(k+1) - x(k+1)$. Assuming that A is full-rank and the pair $(A, \mathcal{C})$ is observable, a steady-state DKF is achieved with error covariance matrices $\{P_i\}_{i \in [n]} = \{\lim_{k \to \infty} P_i(k) \mid i \in [n]\}$. The covariance matrices $\{P_i\}_{i \in [n]}$ satisfy the following equation,

$$P_i = \left(\frac{1}{d_i}\sum_{j \in N_i}(AP_jA^\top + BQB^\top)^{-1} + C_i^\top R^{-1} C_i\right)^{-1} \tag{4}$$

Dutta *et al.* [3] proved the convergence of the covariance matrices $\{P_i(k)\}_{i \in [n]}$ in connected graphs when initialized as zero matrices. Kar *et at.* [12] proved the convergence of covariance matrices using probability theory and Marelli *et al*. [13] performed convergence analysis on a modified DKF which had one prediction/update step at each time point. Note that the results above are based on a static underlying graph, i.e. $\mathcal{G}(k) = \mathcal{G}, \forall k$. We extend the distributed estimator into the case where the underlying graph is time-varying as the covariance matrices may not converge. In this case, a sequence of covariance matrices are generated as follows,

$$P_i(k+1) = \left(\frac{1}{|N_i(k)|}\sum_{j \in N_i(k)}(AP_j(k)A^\top + BQB^\top)^{-1} + C_i^\top R^{-1} C_i\right)^{-1} \tag{5}$$

$$M_j(k) = AP_j(k)A^\top + BQB^\top \tag{6}$$

With the covariance matrices $\{P_i(k)\}_{i \in [n], k \geq 0}$ in (5), the DKF presented in (3) can be rearranged as

$$\begin{aligned} &\frac{1}{|N_i(k)|}\sum_{j \in N_i(k)} M_j^{-1}(\hat{x}_i(k+1) - A\hat{x}_j(k)) \\ &= C_i^\top R^{-1}[y_i(k+1) - C_i\hat{x}_i(k+1)], \end{aligned}$$

where $M_j := M_j(k) = AP_j(k)A^\top + BQB^\top$ for a nonstatic underlying graph. The expression above implies that $\hat{x}_i(k+1)$ is the unique minimizer of the following optimization problem

$$\min_{x_i} \frac{1}{|N_i(k)|}\sum_{j \in N_i(k)} \|x_i - A\hat{x}_j(k)\|^2_{M_j^{-1}} + \|y_i(k+1) - C_i x_i\|^2_{R^{-1}}$$

The above optimization form of DKF reveals the relation of local measurement $y_i(k+1)$ and the information obtained from neighbors. It also shows that the estimation result is sensitive to sensor attacks since any twisted measurement enlarges the objective value by its square form in y_i. To make the optimization-based estimator more robust to attacks, a commonly used strategy is to use optimization with l_1 norm on the terms affected by attacks [14]. We provide the following distributed estimator

$$\hat{x}_i(k+1) = \arg\min_{x_i} f_i(x_i; y_i(k+1), \{\hat{x}_i(k)\}_{i\in[n]})$$

For succinctness, we use $f_i(x_i)$ to denote the loss of node i. Two versions of the loss function f_i are given as follows

$$\begin{aligned} f_i(x_i) = &\|y_i(k{+}1) - C_i x_i - a_i\|^2_{R^{-1}} + \lambda\|a_i\|_1 \\ &+ \frac{1}{d_i(k)} \sum_{j\in N_i(k)} \|x_i - A\hat{x}_j(k)\|^2_{M_j^{-1}} \end{aligned} \tag{7}$$

$$\begin{aligned} f_i(x_i) = &\frac{\lambda}{2} \frac{\|y_i(k{+}1) - C_i x_i\|^2_{R^{-1}}}{\|y_i(k{+}1) - C_i A\hat{x}_i(k)\|_{R^{-1}}} \\ &+ \frac{1}{d_i(k)} \sum_{j\in N_i(k)} \|x_i - A\hat{x}_j(k)\|^2_{M_j^{-1}}, \end{aligned} \tag{8}$$

where a_i is the attack vector occurred at time $k{+}1$. The penalty term $\lambda\|a_i\|_1$ of the first loss function (7) measures the impact of the sensor attack and makes the estimator robust. Equation (7) can be reduced to a LASSO problem [15] minimizing $\|Y - A\theta\| + \lambda\|\theta\|_1$ w.r.t θ, and has already been broadly studied. Compared with (7), the regularization term of λ in (8) has a similar effect of robustness. The parameter λ in (8) also connects the local estimation and the information from its neighbors, a smaller λ will lower the bias caused by the attack but degenerate the convergence rate as the estimation relies more on neighbors' estimation information.

IV. Robust Distributed Estimator with Information Consensus

Olfati-Saber *et al.* [16] showed that the linear system

$$x_i(k+1) = x_i(k) + \sum_{j\in N_i} a_{ij}(x_j(k) - x_i(k))$$

was a distributed consensus algorithm that guaranteed convergence to a collective decision via local interactions among neighbors. As the underlying graph is undirected ($a_{ij} = a_{ji}, \forall i,j$), $\sum_i x_i(k+1) = \sum_i x_i(k)$ and the sum of state of all nodes is invariant. This specific invariance property leads to a type of consensus algorithms called *average-consensus* [17].

We aim to design a distributed estimator in reaching a consensus via local communication with their neighbors. Reaching a consensus here means each local node and its neighbors reach an agreement, i.e. $x_i = x_j, \forall j \in N_i(k)$. We consider the consensus-based robust estimator constructed as the following optimization problem,

$$\begin{aligned} &\min_{\{x_i\}_{i\in[n]}} \sum_{i=1}^n f_i(x_i) \\ &\text{s.t. } x_i = x_j, \quad \forall i\in[n], \forall j \in N_i(k) \end{aligned} \tag{9}$$

where $f_i(x_i)$ is given by (7) or (8).

To solve (9), we introduce auxiliary variables z_i and the problem can be rewritten as

$$\begin{aligned} &\min_{\{x_i\}_{i\in[n]}} \sum_{i=1}^n f_i(x_i) \\ &\text{s.t. } x_i - z_j = 0, \quad \forall i\in[n], \forall j \in N_i(k), \end{aligned} \tag{10}$$

Let v_{ij} be the dual variable of $x_i - z_j$ for all $i,j \in \mathcal{V}$. The augmented Lagrangian (AL) function of (10) is given as follows,

$$L_\rho(\mathbf{x},\mathbf{z};\mathbf{v}) = \sum_{i=1}^n \Big(f_i(x_i) + \frac{\rho}{2} \sum_{j\in N_i(k)} \|x_i - z_j - \frac{1}{\rho} v_{ij}\|^2 \Big),$$

where $\mathbf{x} = (x_1; \cdots; x_n), \mathbf{z} = (z_1;\cdots;z_n), \mathbf{v} = (v_{ij})_{n\times n}$ The corresponding Alternating Direction Method of Multipliers (ADMM) [18] algorithm derived from $L_\rho(\mathbf{x},\mathbf{z};\mathbf{v})$ has the following updates,

$$x_i^{l+1} = \arg\min_{x_i} L_\rho(\mathbf{x}, \mathbf{z}^l; \mathbf{v}^l) \tag{11}$$

$$z_i^{l+1} = \frac{1}{\rho|N_i(k)|} \sum_{j\in N_i(k)} (\rho x_j^{l+1} - v_{ij}^l) \tag{12}$$

$$v_{ij}^{l+1} = v_{ij}^l - \rho(x_i^{l+1} - z_j^{l+1}). \tag{13}$$

The inner iterations (11)–(13) have good performance with a convergence guarantee towards a central solution [19]. One should notice that the x-update in (11) depends on the choice of the loss function f_i. We present in Lemma 2 the solution of (11) when $f_i(x_i)$ in (8) is taken.

Lemma 2: Consider the optimization problem of equation (11) with loss function $f_i(x_i)$ in (8). The solution x_i^{l+1} is based on $y_i(k), \{\hat{x}_i(k)\}_{i\in[n]}$ and has the following expression

$$\begin{aligned} x_i^{l+1} = &\left(\frac{\lambda C_i^\top R^{-1} C_i}{\|y_i(k{+}1) - C_i A\hat{x}_{i(k)}\|_{R^{-1}}} + \frac{2}{|N_i(k)|} \sum_{j\in N_i(k)} M_j^{-1} + \rho|N_i(k)| I \right)^{-1} \\ &\times \Bigg(\frac{2}{|N_i(k)|} \sum_{j\in N_i(k)} M_j^{-1} A\hat{x}_j(k) + \rho \sum_{j\in N_i(k)} z_j^l + \sum_{j\in N_i(k)} v_{ij}^l \\ &\qquad + \frac{\lambda C_i^\top R^{-1} y_i(k+1)}{\|y_i(k{+}1) - C_i A\hat{x}_{i(k)}\|_{R^{-1}}} \Bigg) \end{aligned}$$

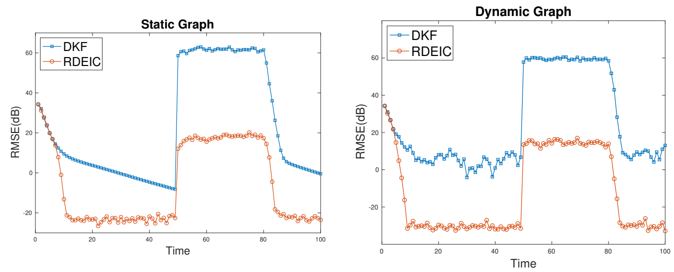
Based on the consensus optimization problem (9), we propose Robust Distributed Estimator with Information Consensus (RDEIC). The RDEIC is summarized in Algorithm 1. We provide two versions of the loss functions in (7) and (8). The updates of the loss function (8) is concluded in Lemma 2.

Algorithm 1 RDEIC

Input: Covariance matrices $\{P_i\}_{i\in[n]}, \{M_i\}_{i\in[n]}$, sensor measurement $\{y_i\}$
Process: Initialize $\{\hat{x}_i(0)\}_{i\in[n]}$
For $k = 1, 2, \cdots$
Update: $f_i(x_i) \leftarrow f_i(x_i; y_i(k+1), \{\hat{x}_i(x)\}_{i\in[n]}), \forall i$
Inner iteration: For $l = 1, \cdots, L$
Repeat (11)–(13) to produce $\{x_i^L\}_{i\in[n]}$
Update: $\hat{x}_i(k+1) \leftarrow x_i^L, \forall i$
Output: State estimation $\{\hat{x}_i(k)\}_{i\in[n]}$.

V. Simulation Results

To validate the resiliency of RDEIC, we run simulations with MATLAB and compare the performance of RDEIC against DKF. The simulation is conducted for $t = 100$ simulation time units with a platoon of 5-vehicle. Each vehicle



(a) RMSE of distance estimation error with static graph.

(b) RMSE of distance estimation error with time-varying graph.

Fig. 1. Performance Comparison between RDEIC and DKF.

is equipped with 4 sensors and the state vector of each vehicle is $x^{(i)} = [d^{(i)}, v^{(i)}, a^{(i)}, u^{(i)}] \in \mathbb{R}^4$, where $d^{(i)}$, $v^{(i)}$, and $a^{(i)}$ are the distance, the velocity, and the acceleration of the i-th vehicle, and $u^{(i)}$ is the control input to the vehicle. The dimension of x is quadruple the number of vehicles, i.e., $x_i \in \mathbb{R}^{20}$. A and $\{C_i\}_{i=1}^n$ are generated based on [6]. The communication graph is randomly generated at each time step. During $t = (50, 80)$, Vehicle 2-4 are compromised and random bias is added to the sensor measurements. The parameters are set as $Q = 0.01\mathbf{I}$, $R = 0.1\mathbf{I}$, $\lambda = 100$ and $\rho = 1$. For performance analysis, we perform $M = 10$ Monte Carlo runs with randomly generated x_i and y_i and use the root mean square error (RMSE) at each time step k as the performance evaluation metric:

$$RMSE = \sqrt{\frac{\sum_{m=1}^{M} \sum_{i=1}^{n} e_{i,k,m}^2}{Mn}}$$

in which $e_{i,k,m}$ is the absolute difference between the true and estimated distance $d^{(i)}$ at time k in Monte Carlo run m.

The performance of RDEIC and DKF is presented in Fig. 1. As can be seen, RDEIC outperforms DKF no matter whether there are attacks or whether the graph is varying. Specifically, the RMSE of RDEIC during attack-free period is less than -20dB, i.e., less than 0.01, with both static graph and time-varying graph. During attacks, the RMSE of DKF exceeds 60dB, which confirms the claim made in [3] that DKF is not resilient to attacks. On the other hand, the RMSE of RDEIC is about 15dB, which shows the resiliency of the proposed algorithm. Besides, in the static graph situation, the estimation error of RDEIC after attack quickly converges to the same level as that before attacks. On the contrary, DKF goes through a longer time to reach a converge.

VI. Conclusion

In this paper, we propose Robust Distributed Estimator with Information Consensus (RDEIC) which performs robustly against sensor measurement attacks. RDEIC is designed to work on both static and time-varying communication graphs and we prove the convergence of the covariance matrices under the jointly strong-connectivity assumption. Two loss functions of each local node are provided to construct an information consensus estimation problem. RDEIC is solved in an ADMM fashion. In the future, we plan to consider more practical constraints like time-delay or packet loss in communication in our model.

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